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CONTACT PROBLEMS OF CREEP THEORY

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The development of creep theory, particularly the proof of the theorem on the influence of creep on the state of stress and strain of an isotropic solid [2 and 26] and the solution of the plane contact problem of plasticity theory [5] produced hypotheses for the analysis of contact problems of creep theory taking account of material ageing. The new effective method of solving first and second kind Fredholm integral equations [18 and 19], which permits obtaining solutions if the solution of the corresponding equation with unit right side is known, also played an essential part. Let us note that from the mechanics viewpoint this solution corresponds to the solution of the plane contact problem for the case of pressure of a rigid stamp with a rectilinear base on a half-plane.

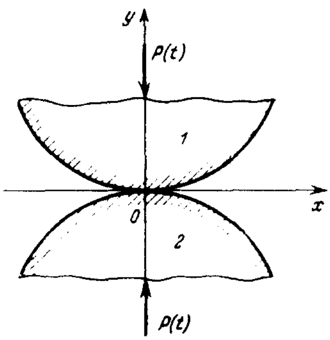

Fig. 1

1. Plane contact problem of creep theory. Prokopovich [26] first studied the plane contact problem of linear creep theory. The known solution of elastisity theory [38] and the fundamental equations [2] of hereditary theory of ageing permitted him to obtain the following Formula :

$$v_i^*(t)=\frac{2}{\pi}[F_i(t)-L_i]\int_S \bar{K}(x,s)\,p^*(s,t)\,ds+C_i(t) \tag{1.1}$$

$$L_i p^*(s,t)=\int_{\tau_1}^{t} K_i(t,\tau)\,p^*(s,\tau)\,d\tau,\qquad F_i(t)=\frac{1-\nu_i^2(t)}{E_i(t)} \tag{1.2}$$

$$K_i(t,\tau)=\frac{\partial}{\partial\tau}\{[1-\nu_i^{*2}(t,\tau)]\,\delta_i(t,\tau)\},\ \bar{K}(x,s)=\ln\frac{1}{|x-s|} \tag{1.3}$$

to determine the vertical displacement v_i^* of the boundary of the i th half-space under plane strain conditions and loaded by normal forces $p^*(x,t)$ applied to the area S.

Here $E_i(t)$ is the modulus of elastic instantaneous strain, $\delta_i(t,\tau)$ the total relative strain, and $\nu_i(t)$ and $\nu_i^*(t,\tau)$ are the lateral expansion coefficients for the elastic strains and creep strains of the ith half-space, respectively (*) .

Let the two bodies (Fig. 1), contacting at a point or along a line and possessing the property of linear creep, be pressed against each other under the effect of external forces whose resultant $P(t)$ is perpendicular to the x-axis and passes through the origin. The relationship which the displacement of points of the contact domain between these bodies should satisfy is, as is known

$$v_1^*(t)+v_2^*(t)=\Delta^*(t)-f_1(x)-f_2(x),\qquad \Delta^*(t)=\Delta_1^*(t)+\Delta_2^*(t) \tag{1.4}$$

Here $\Delta^*(t)$ is the approach of these bodies in the y direction, and $f_1(x)$ and $f_2(x)$ are the equations of the surfaces bounding the first and second bodies.

If it is assumed that there are no friction and adhesion between the compressed bodies, then each of these bodies will experience only a normal pressure $p^*(x,t)$ on the contact section. But the contact domain will usually be small in comparison with the size of the compressed bodies, hence it may be considered that the displacements on the contact section of the compressed bodies will be the same as at the boundary points of two half-planes (upper and lower) subjected to the same norma- pressure $p^*(x,t)$ as are the considered compressed bodies.

Utilizing (1.1) and (1.4), Prokopovich obtained the following integral equation to determine the contact pressure $p^*(x,t)$:

$$\int_S \bar{K}(x,s)\,p^*(s,t)\,ds-\int_{\tau_1}^{t}\int_S \bar{K}(x,s)\,p^*(s,\tau)\,K(t,\tau)\,ds\,d\tau=F(x,t) \tag{1.5}$$

$$K(t,\tau)=\frac{K_1(t,\tau)+K_2(t,\tau)}{F_1(t)+F_2(t)},\qquad F(x,t)=\bar{C}(t)-\frac{f_0(x)}{\theta(t)}$$
$$f_0(x)=f_1(x)+f_2(x),\qquad \theta(t)=A\,[F_1(t)+F_2(t)] \tag{1.6}$$

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*) The total relative strain $\delta(t,\tau)$ observed at time t and caused a unit stress applied at some age of the material τ, is composed of the elastic-instantaneous strain and the creep strain and is defined by the dependence

$$\delta(t,\tau)=\frac{1}{E(\tau)}+C(t,\tau)$$

where $E(\tau)$ is the variable modulus of instantaneous strain, and $C(t,\tau)$ is the measure of the material creep depending on the age of the material and on the duration of the loading action. The measure of creep is understood to be the relative creep strain of ageing material observed up to the time t and caused by unit stress applied at some age τ.

where A is a known quantity which depends on parameters of the material, and $f_1(x)$ and $f_2(x)$ are equations of the contact surfaces of the first and second bodies, $C(t)$ is a function connected with the total inelastic displacement $\Delta^*(t)$.

The integral Eq. (1.5) permits seeking the intensity of the contact pressure $p^*(x, t)$ as a function of the position of points along the contact width and of the duration of the effect of the loading. This equation may be represented more compactly as:

$$\omega(x, t) - \int_{\tau_1}^{t} K(t, \tau)\, \omega(x, \tau)\, d\tau = F(x, t), \qquad \omega(x, t) = \int_S \overline{K}(x, s)\, p^*(x, t)\, ds \tag{1.7}$$

which permits reduction of the problem of determining the unknown contact pressure $p^*(x, t)$ to the successive solution of interrelated integral Eqs. (1.7).

The first of these Eqs., which $\omega(x, t)$ should satisfy as a function of the time t, takes account of the influence of material creep on the contact pressure distribution $p^*(x, t)$ and is a linear Volterra integral equation of the second kind, which has been investigated in detail in [1, 2, 28 and 33] for various cases of the creep kernel $K(t, \tau)$. If the assumption

$$\nu(\tau) = \nu^*(\tau) = \nu = \text{const} \tag{1.8}$$

is made, this equation takes a form analogous to the equation describing the state of stress of a system comprised of two inhomogeneous elements in the presence of forced strains [28]. If two contiguous bodies possess identical elastic moduli and identical creep measures, then the first Eq. in (1.7) is analogous, under the conditions (1.8), to the equation describing stress relaxation in a homogeneous and isotropic solid [2 and 28].

The second integral Eq. in (1.7), which $p^*(x, t)$ should satisfy as a function of x, is a singular Fredholm integral equation of the first kind with kernel $\overline{K}(x, s)$ (defined by (1.3)) and right side $\omega(x, t)$ which is a solution of the first Volterra integral Eq. (1.7).

If the solution of the first integral Eq. in (1.7) is represented as

$$\omega(x, t) = \gamma(t) - H(t)\, f_0(x) \tag{1.9}$$

and, moreover, the Krein method [19] is utilized to solve the second integral Eq. in (1.7) with the right side (1.9) where the conditions (1.8) are taken into account, then the following Formula

$$p^*(x, t) = \frac{\gamma(t)}{\pi\sqrt{a^2 - x^2}} + H(t)\frac{2}{\pi^2}\int_x^a \frac{u\,du}{\sqrt{u^2 - x^2}}\int_0^u \frac{f_0''(s)}{\sqrt{u^2 - s^2}}\, ds \tag{1.10}$$

is obtained to determine $p^*(x, t)$ in the case of the symmetric contact problem for the two bodies under linear creep conditions taking account of material ageing.

Let us note that the function $H(t)$ which is the solution of the first integral Eq. in (1.7) with right side $1/\theta(t)$, takes account of the influence of elastic instantaneous strains and creep strains of compressed bodies taking account of material ageing, on the contact pressure $p^*(x, t)$ in the considered time period.

The first member in (1.10) is a solution with singularities at the points $x = \pm a$, and is subject to being retained only in the case of a given contact with $2a$; the unknown function $\gamma(t)$ is hence determined from the equilibrium Eq.

$$P(t) = \int_{-a}^{a} p^*(x, t)\, dx \tag{1.11}$$

When the contact width $2a = 2a(t)$ is not given, i.e., the contact between the compressed bodies occurs along smooth surfaces, then the unknown function $\gamma = \gamma(t)$ is determined from the requirement that the first member in (1.10), which is a solution with a singularity, will vanish, i.e., $\gamma(t) = 0$, and the time varying contact width $2a = 2a(t)$ will be determined from the equilibrium Eq. (1.11).

Simple computational formulas are obtained for the case of contact along surfaces of particular form. For example, for contact along the cylindrical surface $f_0(x) = \frac{1}{2}x/R$, when the contact width is not given

$$p^*(x, t) = \frac{H(t)}{\pi R}\left(\frac{2RP(t)}{H(t)} - x^2\right)^{1/2} \tag{1.12}$$

Formulas to determine $p^*(x, t)$ in the case of the antisymmetric contact problem of two bodies under linear creep conditions with material ageing taken into account are presented by an analogous means in [26].

It was later shown in [25] that the solution of the plane contact problem of linear creep theory taking account of material ageing for two symmetrically disposed contact sections $(-a \leqslant x \leqslant -b,\ \ b \leqslant x \leqslant a)$ reduces to the solution of the integral Eqs. (1.7), wherein the second of these Eqs. (1.7) has the kernel

$$\overline{K}(x, s) = \ln\frac{1}{x^2 - s^2} \tag{1.13}$$

The solution is constructed by using a substitution of variables and introduction of a new function $q(\zeta, t)$ connected with the pressure $p^*(x, t)$ and defined by a formula analogous to (1.10).

Shirinkulov [37] established that the plane contact problem of linear creep theory taking account of material ageing for bodies whose elastic modulus ages with depth according to a power law can also be reduced to the solution of two integral Eqs. of the type (1.7).

In another paper by the same author [36] a solution of the plane contact problem of linear creep theory taking account of friction when coefficients of lateral expansion of the compressed bodies are equal and constant in time, is presented on the basis of hereditary theory of ageing.

The equilibrium of two compressed orthotropic bodies under plane strain and linear creep conditions with ageing of the material taken into account is examined in [35].

Considerably greater difficulties must be overcome in examining the contact problem taking account of nonlinear creep of the material. This is related to the fact that an analysis of the problem of equilibrium of a half-plane (half-space) loaded by a concentrated force $P(t)$ applied to its boundary and the derivation of formulas to determine the displacement of this boundary under the effect of a distributed pressure $p^*(x, t)$ should precede the construction of the solution.

The plane contact problem of nonlinear creep theory was considered in [4]. The fundamental dependence between the strain intensity $\varepsilon(t)$ and the stress intensity $\sigma(t)$ is taken according to the theory of plastic heredity with material ageing taken into account [2 and 30], as

$$K_0 \varepsilon^{\mu}(t) = \sigma(t) - \int_{\tau_1}^{t} \sigma(\tau) K(t, \tau)\, d\tau \tag{1.14}$$

$$\left(K(t, \tau) = \frac{\partial C(t, \tau)}{\partial \tau},\ K_0 > 0,\ 0 < \mu < 1\right)$$

Here $C(t, \tau)$ is the measure of creep, and K_0 and μ are physical constants of the material. The material is assumed incompressible.

We then obtain the following Formula [4]:

$$v^*(t) = A\,[\,(1-L)\,P(t)]^m\,|\,s-x\,|^{1-m} + D(t), \qquad \left(A = \frac{J}{K_0^m}\,,\ m = \frac{1}{\mu}\right) \quad (1.15)$$

to determine the vertical displacement $v^*(t)$ of the half-plane boundary (or of the half-space boundary in the plane strain case) under the effect of a concentrated force $P(t)$ with nonlinear creep and material ageing taken into account. Here x is the moving abscissa of points of the half-plane boundary for which the displacement $v^*(t)$ is determined, L is the Volterra operator with the creep kernel (1.2) or (1.14); s is the abscissa of points of application of the force $P(t)$; J is a known constant dependent on μ.

Furthermore, it has been shown in [4] that if the pressure diagram $p^*(x, t)$ acting on the contact section $S(a \le x \le b)$ is divided into strips of width Δs_j and height $p^*(s_j, t)$ $(j = 1, 2, \ldots, n)$ and the effect of one of these strips (say, the jth with the abscissa $x = s_j$) in the lower half-plane is considered, then the boundary point of this half-plane with arbitrary abscissa x will receive a displacement $v^*(t)$ in the y direction which is determined by using Formulas

$$\bar{v}(t) = [v^*(t) - D(t)]^{\mu}, \qquad \bar{v}(t) = h_j(t)\,p^*(s_j, t)\,\Delta s_j, \quad (1.16)$$

$$h_j(t) = A^{\mu}\,|\,s_j - x\,|^{\mu-1}\,(1-L) \quad (1.17)$$

Henceforth, $\bar{v}(t)$ will be called the generalized displacement of points of the half-plane boundary. It is here important to note that, as follows from (1.17), the generalized displacement $\bar{v}(t)$ in this case depends on the effective force ; it does not hold for any point of the body for the true displacements $v^*(t)$.

Under the simultaneous action of a system of forces $P_j(t) = p^*(s_j, t)\,\Delta s_j$ $(j = 1, 2, \ldots, n)$; (n is the number of strip elements of width Δs_j on the contact section) on the contact section $S\,(a \le x \le b)$, the possibility of representing the generalized displacement $\bar{v}(t)$ as

$$\bar{v}(t) = \sum_{j=1}^{n} h_j(t)\,p^*(s, t)\,\Delta s_j + {\sum_{j,\,l=1}^{n}}' C_{jk}(t)\,p^*(s_j, t)\,p^*(s_k, t)\,p^*(s_j, t)\,\Delta s_j \Delta s_k + \ldots \quad (1.18)$$

by utilizing a Taylor series expansion is proved, where the prime of the summation means that terms with $j = k$ are omitted.

Because of the smallness of the contact section $S\,(a \le x \le b)$, it is possible to limit oneself to the first principal term of the expansion in (1.18) for the generalized displacement $\bar{v}(t)$ with the degree of accuracy which is customary in solving this problem. After having passed to the limit $\Delta s_j \to 0$ this permits the following Formula

$$\bar{v}(t) = A^{\mu}\left[(1-L)\int_S \widetilde{K}(x, s)\,p^*(s, t)\,ds\right], \qquad \widetilde{K}(x, s) = \frac{1}{|\,s-x\,|^{1-\mu}} \quad (1.19)$$

to be obtained to determine the generalized displacements of the ith half-plane boundary caused by the pressure $p^*(x, t)$ acting on the contact section $S(a \le x \le b)$.

Therefore, (1.19) expresses and gives a foundation to a principle for the approximate superposition of generalized displacements $\bar{v}(t)$, which affords the possibility of reducing the solution of the plane contact problem of nonlinear creep theory taking account of material ageing (or the theory of plasticity with power hardening) to the joint solution of two interrelated integral Eqs. of the form

$$\omega(x, t) - \int_{\tau_1}^{t} \omega(x, \tau)\,K(t, \tau)\,d\tau = F(x, t), \qquad \int_S \frac{p^*(s, t)\,ds}{|\,s-x\,|^{1-\mu}} = \omega(x, t) \quad (1.20)$$

$$F(x, t) = [\gamma(t) - f_0(x)]^{\mu}, \quad f_0(x) = \frac{f_1(x) + f_2(x)}{A_1 + A_2} \qquad A_1 = \frac{J_1}{K_{01}{}^m}, \; A_2 = \frac{J_2}{K_{02}{}^m} \quad (1.21)$$

Here J_1, J_2, K_{01}, K_{02} are material constants of the first and second bodies.

Hence, even in a nonlinear formulation based on the physical dependence (1.14) the contact problem of hereditary creep theory is reduced to the successive solution of two interrelated integral Eqs. (1.20). The solution of the first Eq. in (1.20) has been studied well enough for different kernels $K(t, \tau)$ in [1, 2, 26, 30 and 33] and hence, seeking the function $w(x, t)$ is not difficult. The solution of the Fredholm singular integral equation of the first kind (1.20) is constructed by the Krein method [18 and 19], when the contact domain S between the bodies is a strip $-a \leq x \leq a$.

Formula

$$p^*(x, t) = K(\mu)\left\{\frac{a^{\mu}\Phi_1'(a, t, \gamma)}{\sqrt{(a^2 - x^2)^{\mu}}} - \int_x^a \frac{u^{\mu}\, du}{\sqrt{(u^2 - x^2)^{\mu}}} \int_0^u \frac{\omega''(s, t)\, ds}{\sqrt{(u^2 - s^2)^{\mu}}}\right\} \quad (1.22)$$

$$K(\mu) = \frac{1}{2\pi^{3/2}}\Gamma\left(\frac{1-\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)\left(\sin\frac{\pi\mu}{2}\right)^2, \quad \Phi'(a, t, \gamma) = \frac{d}{da}\int_0^a \frac{\omega(s, t)\, ds}{\sqrt{(a^2 - s^2)^{\mu}}}$$

is obtained for the case, say, of the symmetric contact problem of two bodies under nonlinear creep conditions.

Here $\Gamma(z)$ is the Gamma-function.

For a given contact width $2a$ the unknown function $\gamma = \gamma(t)$ is selected so that condition (1.11) is satisfied, which is equivalent to

$$\Phi_1(a, t, \gamma) = \frac{\pi P}{\sin {}^1/_2\pi\mu} \quad (1.23)$$

The quantity $2a = 2a(t)$ is determined from this same equation for a contact width not given: the function $\gamma(t)$ should hence satisfy the condition that the first term in (1.22), which is the solution with singularities, will vanish, i.e., the condition

$$\Phi_1'(a, t, \gamma) = 0 \quad (1.24)$$

The antisymmetric plane contact problem taking account of nonlinear creep has also been studied and solved. It should be noted that the case of arbitrary loading of compressed bodies cannot be obtained by superposition of the above mentioned two cases (symmetric and antisymmetric), but should be solved separately as an independent problem [4].

The equation and the formulas obtained in [4] for $t = \tau_1$ are the solution of the plane contact problem of plasticity theory with power hardeneing of the material [5], and for $\mu = 1$ the solution of the plane contact problem of elasticity theory [38]. For $m = 0$, as has been shown in [4], the pressure under a rigid plane stamp $p^*(x)$, obtained according to (1.22), agrees with the distribution law corresponding to the known Prandtl solution [16].

On the basis of the solution in [4], Manukian [24] studied the problem of impressing a rigid wedge into a half-plane under unsteady creep conditions. Cases for the contact width given and not given were examined and formulas were derived to determine $\gamma(t)$ and $a(t)$.

The plane contact problem of nonlinear creep theory in the presence of friction under steady creep conditions has been examined in [7]. The dependences between the strain rate intensity ϵ and the stress intensity σ are taken as

$$K_0\varepsilon^{\mu} = \sigma \qquad (0 < \mu < 1) \quad (1.25)$$

Here K_0 and μ are material constants.

If it is assumed that one of the bodies is fixed and a normal pressure $p(x)$ and Coulomb friction $q(x) = kp(x)$ act on the contact section between the contacting bodies, and the superposition principle for generalized displacements is utilized, the solution of the problem of nonlinear creep theory taking account of friction reduces to the solution of a Freholm singular integral equation of the first kind of the following form:

$$\int_{-a}^{a} \frac{[a_2 - \operatorname{sign}(s-x)\, a_1]^{\mu}\, p(s)\, ds}{|s-x|^{1-\mu}} = F(x), \qquad F(x) = [\gamma - f_0(x)]^{\mu}, \quad f_0(x) = \frac{f_1(x) + f_2(x)}{2g_1} \tag{1.26}$$

Here a_1, a_2 and g_1 are constants determined in terms of the physical constants of the material K_0 and μ, and the friction coefficient k.

The structure of the kernel of this integral equation necessitated a special development of a method to obtain the solution. For the case of impression of a rigid stamp with a rectilinear base into a half-plane, i.e., when the right side of the singular integral Eq. (1.26) is a constant, an exact solution of this integral equation in closed form has successfully been obtained [7]. By utilizing this solution, Formula

$$p(x) = \frac{\Gamma(^1/_2(3-\mu))\,\Gamma(^1/_2\mu)\,\Gamma(\mu-\rho)\sin\pi(\mu-\rho)}{2a^{1-\mu}\,\Gamma(1-\rho)\sqrt{\pi}}\,\frac{P}{\pi\sqrt{(a^2-x^2)^{\mu}}}\left(\frac{a+x}{a-x}\right)^{1/2\mu-\rho}$$

has been obtained to determine the contact pressure $p(x)$, which contains a constant ρ defined in terms of the parameters a_1, a_2 and μ.

Manukian [24a] has extended the solution of the contact problem taking account of friction to the unsteady creep case described by the dependence (1.14).

2. Three-dimensional contact problem of creep theory. Kakosimidi and Prokopovich [14] studied the three-dimensional contact problem of creep theory in a linear formulation. Upon compliance with condition (1.8) a formula has been obtained to determine the vertical displacements of the boundary of the ith half-space

$$v_i^*(t) = A_i\,[F_i(t) - L_i] \iint_{(S)} \bar{K}(s, \eta)\, p^*(s, \eta, t)\, ds\, d\eta \tag{2.1}$$

$$\left(A_i = \frac{1-\nu_i^2}{\pi}, \quad F_i(t) = \frac{1}{E_i(t)}, \quad \bar{K}(s,\eta) = \frac{1}{\sqrt{(x^2-s^2)+(y^2-\eta^2)^2}}\right)$$

Here the operator L_i is defined by the relationship (1.2).

The unknown contact pressure $p^*(x, y, t)$ is determined from the integral Eq.

$$\iint_{(S)} \bar{K}(\ ,\eta)\, p^*(s,\eta,t)\, ds\, d\eta - \int_{\tau_1}^{t}\iint_{(S)} \bar{K}(s,\eta)\, K(t,\tau)\, p^*(s,\eta,\tau)\, ds\, d\eta\, d\tau = F(x, y, t) \tag{2.2}$$

which is analogous to (1.5) and admits of representation as the two Eqs.

$$\omega(x, y, t) - \int_{\tau_1}^{t} \omega(x, y, \tau)\, K(t, \tau)\, d\tau = F(x, y, t), \tag{2.3}$$

$$\omega(x, y, t) = \iint_{(S)} \bar{K}(s, \eta)\, p^*(s, \eta, t)\, ds\, d\eta$$

Here $K(t, \tau)$ and $F(x, y, t)$ are defined by formulas analogous to (1.6) by taking into account that $f_0 = f_0(x, y) = f_1(x, y) + f_2(x, y)$.

The integral Eqs. (2.3) and (1.7) are completely identical. The second Eq. in (2.3) describes the elastic instantaneous problem for a fixed time t.

The three-dimensional problem of linear creep theory taking account of material ageing was also considered by Predeleanu [39]; the obtained results were applied, in particular, to the solution of the contact problem of two spherical bodies subjected to a constant compressive force.

In [11] Efimov examined the axisymmetric contact problem for linearly viscoelastic bodies. The author expresses the contact pressure in terms of an integral operator acting on some function of the coordinate r and the time t which maps the influence of the loading and unloading. It has been established that the connection between the contact pressure and the radius of the contact circle under repeated unloading does not depend on the complete history of the contact process, but on the appropriate "truncated trajectory" of loading-unloading.

On the basis of ideas developed in [4 and 5], Kuznetsov solved the problem of impression of a rigid stamp in a half-space under nonlinear creep conditions characterized by a physical equation analogous to (1.14), or for power hardening of the material. Analysis of the problem of equilibrium of a half-space taking account of material creep under the effect of a concentrated force $P(t)$, deduction of formulas to determine the displacement of the boundary of this half-space under steady creep conditions under the effect of the distributed pressure $p^*(x, y, t)$ and, finally, the solution of the problem of impression of a stamp in a half-space with power hardening of the material all preceded the construction of the solution of the problem under consideration.

The solution of the problem of impression of a rigid stamp in a half-space for nonlinear creep of the material is based on the possibility of representing the vertical displacements of the half-space boundary by Formula

$$v^*(t) = A\left[(1-L)\iint_{(S)} \widetilde{K}(s, \eta)\, p^*(s, \eta, t)\, ds\, d\eta\right]^m, \quad m = \frac{1}{\mu} \tag{2.4}$$

which is obtained by starting from the superposition principle for the generalized displacements. Here A is a known quantity dependent on the material parameters K_0 and $\mu(0<\mu<1)$, the operator L is determined according to (1.2) and (1.3) or (1.14), and the kernel is

$$\widetilde{K}(s, \eta) = \frac{1}{[\sqrt{(x-s)^2 + (y-\eta)^2}]^{2-\mu}} \tag{2.5}$$

This formula permitted reduction of the problem of determining the unknown contact pressure $p^*(x, y, t)$ to the solution of two integral equations of the type (2.3). The kernels $K(t, \tau)$ and $\widetilde{K}(s, \eta)$ are here taken in accordance with (1.3), (1.14) and (2.5), and $F(x, y, t)$ according to (1.21) taking into account that

$$f_0(x, y) = \frac{f_1(x, y) + f_2(x, y)}{A}$$

and $\gamma(t)$, in the general case, is $\gamma(t) = \Delta^*(t) + \alpha(t)x + \beta(t)y$; three equilibrium conditions for the stamp are used to determine the unknown functions $\Delta^*(t)$, $\alpha(t)$ and $\beta(t)$.

The case of a stamp with a flat elliptical base and the case of an axisymmetric stamp have been considered in [21].

All the solutions obtained in the above mentioned papers for the problems of contact between two bodies under unsteady creep conditions refer to cases when the determination of the contact pressure $p^*(x, t)$ reduces to the successive solution of two interrelated integral equations of type (1.20) or (2.3).

Naturally it turned out to be possible to obtain the solution of contact problems of unsteady creep theory from this kind of system of equations because of definite assumptions on the physical dependence between the stresses and strains which underlies hereditary creep theory.

It is characteristic that according to the solutions obtained above, creep of the material of bodies in contact does not affect the distribution law of the contact stiesses in time, if the contact between these bodies is along a line or a plane, as for example in the impression of rigid stamps in a half-plane or in a half-space under unsteady creep conditions.

Creep of the material of bodies in contact infuences the pressure distribution only when the contact between these bodies is along curved surfaces, where the contact area changes in time under the prolonged action of the loading for an unspecified contact width.

3. Taking account of creep of the base in solving different contact problems. Questions of the analysis of beams on an elastic foundation possessing creep have been examined by Rzhanitsyn [31], Rozovskii [34], Gol'denblat and Nikolaenko [8] and Prokopovich [28].

A solution is presented in these papers for the problem of analyzing beams of a continuous foundation when the beam and foundation materials correspond to the rheological model of a viscoelastic Kelvin body, or are subject to the laws of the linear theory of elastic heredity.

Rozovskii [32] gave a solution of the problem of deformation of a foundation beam taking account of creep of the foundation when the latter is subject to the Winkler bedding coefficient.

In [17] Kiiss presented an analysis of reinforced concrete beams taking account of creep of the concrete and the foundation by starting from the condition that the curvatures of the beam and the foundation surface are equal. To solve the original problem Kiiss used approximate methods. As an application he examined a problem in which the foundation is laminar. Ageing of the beam and foundation materials was not taken into account in solving the problem.

A peculiar contact problem is the problem of the thermally stressed state of a massive concrete block on a rock or pre-laid concrete foundation. The appropriate solution of the plane problem has been expounded in [3]; the authors assumed that an elastic layer is disposed between the foundation and the block. This solution was later expanded by Manukian and Zadoian and applied to circular [23] and rectangular blocks [12] taking account of creep of the concrete. Prokopovich proposed an approximate method of analizing concrete blocks taking account of their elastic properties and the creep of the foundation. The appropriate solution allowed for clarification of the peculiarities of the influence of the relationship between the geometric dimensions of the blocks on its thermally stressed state [27] and was a basis for the creation of a practical computational method [29]. This method permits taking account of: the change in the temperature and humidity regimes, the geometric dimensions of the block, the construction of the foundation, the change in modulus of elastic instantaneous strains, and the stress relaxation because of creep of the concrete. The thermally stressed state of a system of two massive blocks has recently been studied [20].

On the basis of the Prokopovich ideas [26], Kakosimidi [13], using the hereditary theory of ageing, developed an approximate method of analyzing foundation strips and circular slabs [15] on a creep-elastic foundation. To describe the mechanical properties of the foundation, the author used the model of a creep-elastic half-space subject to plane strain. The problem was reduced to solving a Volterra integral equation of the second kind. Taking account of creep of the foundation in the analysis of foundation strips (as well as beams) leads to a growth in the nominal forces, to a noticeable redistribution in the concrete pressures, and an increase in the bending moments.

Erzhanov [9] considered several problems devoted to the investigation of the stress-strain state of underground structures taking account of the creep of rock strata on the basis of the Rabotnov hereditary creep theory. He investigated the stress-strain state of a mountain mass around a reinforced and unreinforced mine and horizontal excavation taking account of creep of the rock.

Erzhanov and Egorov [10] investigated the mechanism of growth of folded structure in the earth's core for tectonic processes by starting from the model of a relaxing viscoelastic body represented by Volterra equations.

Utilizing general principles of thermodynamics, Lapidus [22] obtained, under certain assumptions, a hereditary influence function for rock strate which agrees outwardly with the Kohlrausch-Bronsky function and well with experiment.

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HIGH FREQUENCY OSCILLATIONS OF A THIN ELASTIC LAYER OF VARIABLE THICKNESS

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We investigate shear oscillations of a thin elastic layer $0 \leq z \leq h(x)$ of variable thickness, where $h(x)$ is a sufficiently smooth function. One boundary of this layer is free, while the other is in contact with a nonhomogeneous elastic medium, the contact defined by a boundary condition containing an impedance. Oscillations are of high-frequency

$$\Omega \equiv \frac{\omega h(x)}{b} \gg 1$$

Here ω is the frequency and b denotes the rate of propagation of shear waves. The displacement vector is parallel to the Y-axis.

Solution of the problem is constructed in the form of special asymptotic power series in $\Omega^{-1/3}$. Displacements of the layer are expressed in terms of $h(x)$ and of the properties of the elastic medium in contact with the layer. Expressions are found for the phase and group velocity within the layer. Final formulas are also obtained by another method based on the idea of constructive interference of the volume waves. Radial interpretation of the dependence of wave intensity on the variables x and z and ray tracing method, are used to obtain the decay of perturbations propagating along the layer.